MULTIHOP CONNECTIVITY OF ARBITRARY NETWORKS

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The adjacency (one-hop connectivity) matrix $A = \{a_{ij}\}$ for an N-node network, in which a 1 entry at (i, j) indicates a connection from node i to node j and a 0 entry at $(i,\, j)$ indicates no connection from node i to node j, can be manipulated to obtain the (multihop) connectivity matrix $C = \{c_{ij}\}$, for which the entry at (i, j) lists the minimum number of hops needed to connect node i to node j. The key to understanding this fact is to realize that A is a method for listing the neighbors of each node.

Note that the diagonal elements of the adjacency matrix equal zero, $a_{ii} = 0$ for all i, and for $i \neq j$,

$$a_{ij} = \begin{cases} 1, & \text{link } i \rightarrow j \text{ exists} \\ 0, & \text{link } i \rightarrow j \text{ does not exist} \end{cases} \tag{1}$$

Let us define an intermediate calculation of the connectivity matrix as $C^{(m)} = \left\{ c_{ij}^{(m)} \right\}$, where $c_{ii}^{(m)} = 0$ and

$$c_{ij}^{(m)} = \begin{cases} h, & i \text{ connected to } j \text{ by } h \leq m \text{ hops} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

According to this definition, the adjacency matrix for the network is the first iteration in calculating the connectivity matrix. That is, $C^{(1)} = A$. Now, consider the elements of the square of matrix A, which we denote $A^2 = \left\{ a_{ij}^{(2)} \right\}$. These elements can be written

$$a_{ij}^{(2)} = \sum_{k=1}^{N} a_{ik}\, a_{kj} = \begin{cases} 0, & \text{no path } i \rightarrow k \rightarrow j \text{ exists} \\ > 0, & \text{at least one path } i \rightarrow k \rightarrow j \text{ exists} \end{cases} \tag{3}$$

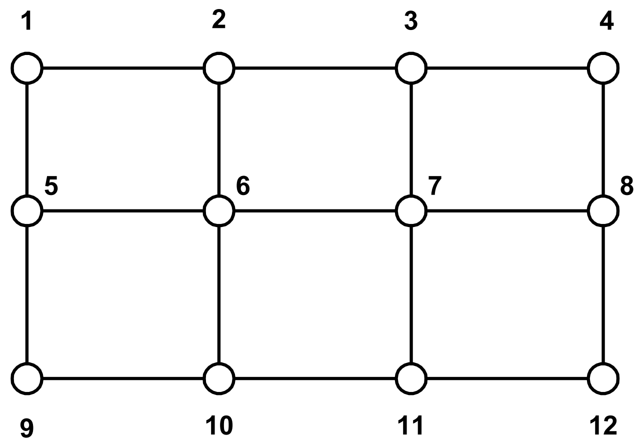
Note that every node that has at least one neighbor will have a two-hop circular path back to itself. Also, there may be more than one two-hop path from i to j, and it is possible for a two-hop path to exist between nodes whose shortest connection is one hop. So, to obtain the second iteration in calculating the connectivity matrix, we first modify $a_{ij}^{(2)}$ as follows:

$$b_{ij}^{(2)} = \begin{cases} 0, & i = j & \text{(eliminate looping paths)} \\ 0, & a_{ij} = 1 & \text{(eliminate 2-hop path when there is already a 1-hop path)} \\ 2, & a_{ij}^{(2)} > 0 & \text{when } i \neq j \text{ and } a_{ij} = 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and then $c_{ij}^{(2)}$ is found as

$$c_{ij}^{(2)} = a_{ij} + b_{ij}^{(2)} = c_{ij}^{(1)} + b_{ij}^{(2)} \tag{5}$$

For example, consider the mesh network diagrammed in the following figure:



The adacency matrix for this 3×4-node mesh network is given by

$$A = C^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

and the square of this matrix is given by

$$A^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 4 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 & 4 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \tag{7}$$

Applying (4) to (7), we obtain the matrix of only two-hop connections:

$$B^{(2)} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \tag{8}$$

Adding (8) to (6) gives the intermediate calculation of the connectivity matrix for $m = 2$:

$$C^{(2)} = C^{(1)} + B^{(2)} = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix} \tag{8}$$

The general rule for the intermediate calculations may be stated as follows:

$$c_{ij}^{(m)} = c_{ij}^{(m-1)} + b_{ij}^{(m)}, \quad m \geq 2 \tag{9}$$

where

$$b_{ij}^{(m)} = \begin{cases} 0, & i = j \\ 0, & c_{ij}^{(m-1)} > 0 \\ m, & \sum_{k=1}^{N} c_{ik}^{(m-1)} a_{kj} > 0 \text{ when } i \neq j \text{ and } c_{ij}^{(m-1)} = 0 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

The third line of the right-hand side of (10) suggests taking the product of $C^{(m-1)}$ and A. However, a more efficient algorithm is the following:

ALGORITHM TO COMPUTE CONNECTIVITY MATRIX FROM ADJACENCY MATRIX

- Define two $N \times N$ matrices B and C, in addition to the $N \times N$ adjacency matrix A.
- Initially, set $C = A$ and $B = 0$.
- For $m = 2$ to the longest hop distance (or until the update matrix B equals zero):
  - For all node pairs $(i, j) = (1, 1)$ to (N, N):
    - If $i = j$, skip to the next node pair
    - If $c_{ij} > 0$, skip to the next node pair
    - For $k = 1$ to N
      - If $c_{ik} > 0$ and $a_{kj} > 0$ for some k,
        - Set $b_{ij} = m$
        - Exit the loop and go to the next node pair (i, j)
  - Set $C \leftarrow C + B$, then $B = 0$
- At the end of these calculations, C equals the connectivity matrix and the sum of all the elements of C, divided by $N(N-1)$, equals the average hop distance.